Chapter 11

Introduction To Three Dimensional Geometry

Exercise 11.1

Question 1:

A point is on the *x*-axis. What are its *y*-coordinates and *z*-coordinates?

Answer:

If a point is on the *x*-axis, then its *y*-coordinates and *z*-coordinates are zero.

Question 2:

A point is in the XZ-plane. What can you say about its *y*-coordinate?

Answer:

If a point is in the XZ plane, then its *y*-coordinate is zero.

Question 3:

Name the octants in which the following points lie:

(1, 2, 3), (4, −2, 3), (4, −2, −5), (4, 2, −5), (−4, 2, −5), (−4, 2, 5),

(−3, −1, 6), (2, −4, −7)

Answer:

The *x*-coordinate, *y*-coordinate, and *z*-coordinate of point (1, 2, 3) are all positive. Therefore, this point lies in octant **I**.

The *x*-coordinate, *y*-coordinate, and *z*-coordinate of point (4, −2, 3) are positive, negative, and positive respectively. Therefore, this point lies in octant **IV**.

The *x*-coordinate, *y*-coordinate, and *z*-coordinate of point (4, −2, −5) are positive, negative, and negative respectively. Therefore, this point lies in octant **VIII**.

The *x*-coordinate, *y*-coordinate, and *z*-coordinate of point (4, 2, −5) are positive, positive, and negative respectively. Therefore, this point lies in octant **V**.

The *x*-coordinate, *y*-coordinate, and *z*-coordinate of point (−4, 2, −5) are negative, positive, and negative respectively. Therefore, this point lies in octant **VI**.

The *x*-coordinate, *y*-coordinate, and *z*-coordinate of point (–4, 2, 5) are negative, positive, and positive respectively. Therefore, this point lies in octant **II**.

The *x*-coordinate, *y*-coordinate, and *z*-coordinate of point (–3, –1, 6) are negative, negative, and positive respectively. Therefore, this point lies in octant **III**.

The *x*-coordinate, *y*-coordinate, and *z*-coordinate of point (2, –4, –7) are positive, negative, and negative respectively. Therefore, this point lies in octant **VIII**.

Question 4:

Fill in the blanks:

Answer:

(i) The *x*-axis and *y*-axis taken together determine a plane known as $\underline{\text{XY} - \text{plane}}$.

(ii) The coordinates of points in the XY-plane are of the form $\underline{(x, y, 0)}$.

(iii) Coordinate planes divide the space into $\underline{\text{eight}}$ octants.

Exercise 11.2

Question 1:

Find the distance between the following pairs of points:

(i) (2, 3, 5) and (4, 3, 1) (ii) (–3, 7, 2) and (2, 4, –1)

(iii) (–1, 3, –4) and (1, –3, 4) (iv) (2, –1, 3) and (–2, 1, 3)

Answer:

The distance between points P(x_1, y_1, z_1) and P(x_2, y_2, z_2) is given by $\text{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

(i) Distance between points (2, 3, 5) and (4, 3, 1)

$$= \sqrt{(4-2)^2 + (3-3)^2 + (1-5)^2}$$
$$= \sqrt{(2)^2 + (0)^2 + (-4)^2}$$
$$= \sqrt{4+16}$$
$$= \sqrt{20}$$
$$= 2\sqrt{5}$$

(ii) Distance between points (−3, 7, 2) and (2, 4, −1)

$$= \sqrt{(2+3)^2 + (4-7)^2 + (-1-2)^2}$$
$$= \sqrt{(5)^2 + (-3)^2 + (-3)^2}$$
$$= \sqrt{25+9+9}$$
$$= \sqrt{43}$$

(iii) Distance between points (−1, 3, −4) and (1, −3, 4)

$$= \sqrt{(1+1)^2 + (-3-3)^2 + (4+4)^2}$$
$$= \sqrt{(2)^2 + (-6)^3 + (8)^2}$$
$$= \sqrt{4+36+64} = \sqrt{104} = 2\sqrt{26}$$

(iv) Distance between points (2, −1, 3) and (−2, 1, 3)

$$= \sqrt{(-2-2)^2 + (1+1)^2 + (3-3)^2}$$
$$= \sqrt{(-4)^2 + (2)^2 + (0)^2}$$
$$= \sqrt{16+4}$$
$$= \sqrt{20}$$

$$= 2\sqrt{5}$$

Question 2:

Show that the points (−2, 3, 5), (1, 2, 3) and (7, 0, −1) are collinear.

Answer:

Let points (−2, 3, 5), (1, 2, 3), and (7, 0, −1) be denoted by P, Q, and R respectively.

Points P, Q, and R are collinear if they lie on a line.

$$PQ = \sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2}$$
$$= \sqrt{(3)^2 + (-1)^2 + (-2)^2}$$
$$= \sqrt{9+1+4}$$
$$= \sqrt{14}$$

$$QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2}$$
$$= \sqrt{(6)^2 + (-2)^2 + (-4)^2}$$
$$= \sqrt{36+4+16}$$
$$= \sqrt{56}$$
$$= 2\sqrt{14}$$

$$PR = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2}$$
$$= \sqrt{(9)^2 + (-3)^2 + (-6)^2}$$
$$= \sqrt{81+9+36}$$
$$= \sqrt{126}$$
$$= 3\sqrt{14}$$

Here, PQ + QR $= \sqrt{14} + 2\sqrt{14} = 3\sqrt{14}$ = PR

Hence, points P(−2, 3, 5), Q(1, 2, 3), and R(7, 0, −1) are collinear.

Question 3:

Verify the following:

(i) (0, 7, −10), (1, 6, −6) and (4, 9, −6) are the vertices of an isosceles triangle.

(ii) (0, 7, 10), (−1, 6, 6) and (−4, 9, 6) are the vertices of a right angled triangle.

(iii) (−1, 2, 1), (1, −2, 5), (4, −7, 8) and (2, −3, 4) are the vertices of a parallelogram.

Answer:

(i) Let points (0, 7, −10), (1, 6, −6), and (4, 9, −6) be denoted by A, B, and C respectively.

$$AB=\sqrt{(1-0)^2+(6-7)^2+(-6+10)^2}$$
$$=\sqrt{(1)^2+(-1)^2+(4)^2}$$
$$=\sqrt{1+1+16}$$
$$=\sqrt{18}$$
$$=3\sqrt{2}$$

$$BC=\sqrt{(4-1)^2+(9-6)^2+(-6+6)^2}$$
$$=\sqrt{(3)^2+(3)^2}$$
$$=\sqrt{9+9}=\sqrt{18}=3\sqrt{2}$$

$$CA=\sqrt{(0-4)^2+(7-9)^2+(-10+6)^2}$$
$$=\sqrt{(-4)^2+(-2)^2+(-4)^2}$$
$$=\sqrt{16+4+16}=\sqrt{36}=6$$

Here, AB = BC ≠ CA

Thus, the given points are the vertices of an isosceles triangle.

(ii) Let (0, 7, 10), (−1, 6, 6), and (−4, 9, 6) be denoted by A, B, and C respectively.

$$AB=\sqrt{(-1-0)^2+(6-7)^2+(6-10)^2}$$
$$=\sqrt{(-1)^2+(-1)^2+(-4)^2}$$
$$=\sqrt{1+1+16}=\sqrt{18}$$
$$=3\sqrt{2}$$

$$BC=\sqrt{(-4+1)^2+(9-6)^2+(6-6)^2}$$
$$=\sqrt{(-3)^2+(3)^2+(0)^2}$$
$$=\sqrt{9+9}=\sqrt{18}$$
$$=3\sqrt{2}$$

$$CA = \sqrt{(0+4)^2 + (7-9)^2 + (10-6)^2}$$
$$= \sqrt{(4)^2 + (-2)^2 + (4)^2}$$
$$= \sqrt{16+4+16}$$
$$= \sqrt{36}$$
$$= 6$$

Now, $AB^2 + BC^2 = \left(3\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2 = 18 + 18 = 36 = AC^2$

Therefore, by Pythagoras theorem, ABC is a right triangle.

Hence, the given points are the vertices of a right-angled triangle.

(iii) Let (−1, 2, 1), (1, −2, 5), (4, −7, 8), and (2, −3, 4) be denoted by A, B, C, and D respectively.

$$AB = \sqrt{(1+1)^2 + (-2-2)^2 + (5-1)^2}$$
$$= \sqrt{4+16+16}$$
$$= \sqrt{36}$$
$$= 6$$

$$BC = \sqrt{(4-1)^2 + (-7+2)^2 + (8-5)^2}$$
$$= \sqrt{9+25+9} = \sqrt{43}$$

$$CD = \sqrt{(2-4)^2 + (-3+7)^2 + (4-8)^2}$$
$$= \sqrt{4+16+16}$$
$$= \sqrt{36}$$
$$= 6$$

$$DA = \sqrt{(-1-2)^2 + (2+3)^2 + (1-4)^2}$$
$$= \sqrt{9+25+9} = \sqrt{43}$$

Here, AB = CD = 6, BC = AD = $\sqrt{43}$

Hence, the opposite sides of quadrilateral ABCD, whose vertices are taken in order, are equal.

Therefore, ABCD is a parallelogram.

Hence, the given points are the vertices of a parallelogram.

Question 4:

Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, −1).

Answer:

Let P (x, y, z) be the point that is equidistant from points A(1, 2, 3) and B(3, 2, −1).

Accordingly, PA = PB

$$\Rightarrow PA^2 = PB^2$$
$$\Rightarrow (x-1)^2 + (y-2)^2 + (z-3)^2 = (x-3)^2 + (y-2)^2 + (z+1)^2$$

$\Rightarrow x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9 = x^2 - 6x + 9 + y^2 - 4y + 4 + z^2 + 2z + 1$

$\Rightarrow -2x - 4y - 6z + 14 = -6x - 4y + 2z + 14$

$\Rightarrow -2x - 6z + 6x - 2z = 0$

$\Rightarrow 4x - 8z = 0$

$\Rightarrow x - 2z = 0$

Thus, the required equation is $x - 2z = 0$.

Question 5:

Find the equation of the set of points P, the sum of whose distances from A (4, 0, 0) and B (−4, 0, 0) is equal to 10.

Answer:

Let the coordinates of P be (x, y, z).

The coordinates of points A and B are (4, 0, 0) and (−4, 0, 0) respectively.

It is given that PA + PB = 10.

$$\Rightarrow \sqrt{(x-4)^2 + y^2 + z^2} + \sqrt{(x+4)^2 + y^2 + z^2} = 10$$
$$\Rightarrow \sqrt{(x-4)^2 + y^2 + z^2} = 10 - \sqrt{(x+4)^2 + y^2 + z^2}$$

On squaring both sides, we obtain

$$\Rightarrow (x-4)^2 + y^2 + z^2 = 100 - 20\sqrt{(x+4)^2 + y^2 + z^2} + (x+4)^2 + y^2 + z^2$$
$$\Rightarrow x^2 - 8x + 16 + y^2 + z^2 = 100 - 20\sqrt{x^2 + 8x + 16 + y^2 + z^2} + x^2 + 8x + 16 + y^2 + z^2$$
$$\Rightarrow 20\sqrt{x^2 + 8x + 16 + y^2 + z^2} = 100 + 16x$$
$$\Rightarrow 5\sqrt{x^2 + 8x + 16 + y^2 + z^2} = (25 + 4x)$$

On squaring both sides again, we obtain

$25(x^2 + 8x + 16 + y^2 + z^2) = 625 + 16x^2 + 200x$

$\Rightarrow 25x^2 + 200x + 400 + 25y^2 + 25z^2 = 625 + 16x^2 + 200x$

$\Rightarrow 9x^2 + 25y^2 + 25z^2 - 225 = 0$

Thus, the required equation is $9x^2 + 25y^2 + 25z^2 - 225 = 0$.

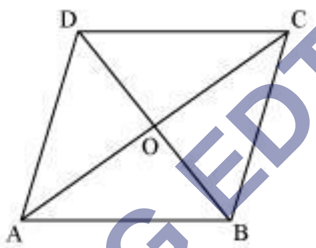
Miscellaneous Exercise

Question 1:

Three vertices of a parallelogram ABCD are A (3, −1, 2), B (1, 2, −4) andC (−1, 1, 2). Find the coordinates of the fourth vertex.

Answer:

The three vertices of a parallelogram ABCD are given as A (3, −1, 2), B (1, 2, −4), and C (−1, 1, 2). Let the coordinates of the fourth vertex be D (*x*, *y*, *z*).



We know that the diagonals of a parallelogram bisect each other.

Therefore, in parallelogram ABCD, AC and BD bisect each other.

∴Mid-point of AC = Mid-point of BD

$$\Rightarrow\left(\frac{3-1}{2},\frac{-1+1}{2},\frac{2+2}{2}\right)=\left(\frac{x+1}{2},\frac{y+2}{2},\frac{z-4}{2}\right)$$

$$\Rightarrow(1,0,2)=\left(\frac{x+1}{2},\frac{y+2}{2},\frac{z-4}{2}\right)$$

$$\Rightarrow\frac{x+1}{2}=1,\ \frac{y+2}{2}=0,\ \text{and}\ \frac{z-4}{2}=2$$

⇒ $x = 1$, $y = -2$, and $z = 8$

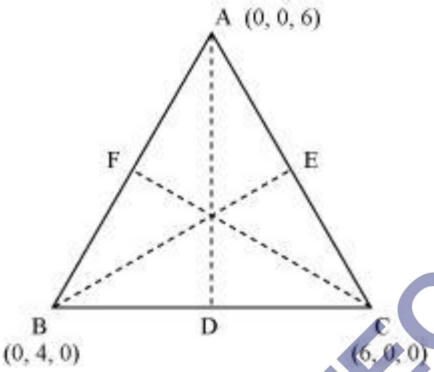
Thus, the coordinates of the fourth vertex are (1, −2, 8).

Question 2:

Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0, 4, 0) and (6, 0, 0).

Answer:

Let AD, BE, and CF be the medians of the given triangle ABC.



Since AD is the median, D is the mid-point of BC.

∴Coordinates of point D = $\left(\frac{0+6}{2},\frac{4+0}{2},\frac{0+0}{2}\right)$ = (3, 2, 0)

$$AD=\sqrt{(0-3)^2+(0-2)^2+(6-0)^2}=\sqrt{9+4+36}=\sqrt{49}=7$$

Since BE is the median, E is the mid-point of AC.

$$\therefore \text{Coordinates of point E}=\left(\frac{0+6}{2},\frac{0+0}{2},\frac{6+0}{2}\right)=(3,0,3)$$

$$BE=\sqrt{(3-0)^2+(0-4)^2+(3-0)^2}=\sqrt{9+16+9}=\sqrt{34}$$

Since CF is the median, F is the mid-point of AB.

$$\therefore \text{Coordinates of point F}=\left(\frac{0+0}{2},\frac{0+4}{2},\frac{6+0}{2}\right)=(0,2,3)$$

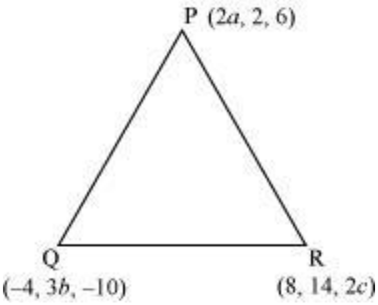
$$\text{Length of CF} = \sqrt{(6-0)^2+(0-2)^2+(0-3)^2} = \sqrt{36+4+9} = \sqrt{49} = 7$$

Thus, the lengths of the medians of ΔABC are $7, \sqrt{34}, \text{ and } 7$.

Question 3:

If the origin is the centroid of the triangle PQR with vertices P (2*a*, 2, 6), Q (−4, 3*b*, −10) and R (8, 14, 2*c*), then find the values of *a*, *b* and *c*.

Answer:



It is known that the coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

Therefore, coordinates of the centroid of ΔPQR $=\left(\frac{2a-4+8}{3}, \frac{2+3b+14}{3}, \frac{6-10+2c}{3}\right)=\left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$

It is given that origin is the centroid of ΔPQR.

$$\therefore (0,0,0)=\left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$$

$$\Rightarrow \frac{2a+4}{3}=0, \frac{3b+16}{3}=0 \text{ and } \frac{2c-4}{3}=0$$

$$\Rightarrow a=-2, b=-\frac{16}{3} \text{ and } c=2$$

Thus, the respective values of *a*, *b*, and *c* are $-2, -\frac{16}{3}, \text{ and } 2.$

Question 4:

If A and B be the points (3, 4, 5) and (−1, 3, −7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where *k* is a constant.

Answer:

The coordinates of points A and B are given as (3, 4, 5) and (−1, 3, −7) respectively.

Let the coordinates of point P be (x, y, z).

On using distance formula, we obtain

$$\begin{aligned}
PA^2 &= (x-3)^2 + (y-4)^2 + (z-5)^2 \\
&= x^2 + 9 - 6x + y^2 + 16 - 8y + z^2 + 25 - 10z \\
&= x^2 - 6x + y^2 - 8y + z^2 - 10z + 50 \\
PB^2 &= (x+1)^2 + (y-3)^2 + (z+7)^2 \\
&= x^2 + 2x + y^2 - 6y + z^2 + 14z + 59
\end{aligned}$$

Now, if $PA^2 + PB^2 = k^2$, then

$$\begin{aligned}
&(x^2 - 6x + y^2 - 8y + z^2 - 10z + 50) + (x^2 + 2x + y^2 - 6y + z^2 + 14z + 59) = k^2 \\
&\Rightarrow 2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 = k^2 \\
&\Rightarrow 2(x^2 + y^2 + z^2 - 2x - 7y + 2z) = k^2 - 109 \\
&\Rightarrow x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2 - 109}{2}
\end{aligned}$$

Thus, the required equation is $x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2 - 109}{2}$.