# Chapter 7

# Binomial Theorem

## Exercise 7.1

**Question** 1:

Expand the expression $(1-2x)^5$

**Answer**:

By using Binomial Theorem, the expression $(1-2x)^5$ can be expanded as

$$
\begin{aligned}
&(1-2x)^5 \\
&= {}^5C_0(1)^5 - {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)^1(2x)^4 - {}^5C_5(2x)^5 \\
&= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - (32x^5) \\
&= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5
\end{aligned}
$$

**Question** 2:

Expand the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

**Answer**:

By using Binomial Theorem, the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ can be expanded as

$$
\begin{aligned}
\left(\frac{2}{x}-\frac{x}{2}\right)^5 &= {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 \\
&\quad - {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5 \\
&= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32} \\
&= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}
\end{aligned}
$$

**Question** 3:

Expand the expression $(2x-3)^6$

**Answer**:

By using Binomial Theorem, the expression $(2x - 3)^6$ can be expanded as

$$
\begin{aligned}
(2x-3)^6 &= {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3 \\
&\quad + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6 \\
&= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) \\
&\quad + 15(4x^2)(81) - 6(2x)(243) + 729 \\
&= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729
\end{aligned}
$$

**Question** 4:

Expand the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

**Answer**:

By using Binomial Theorem, the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ can be expanded as

$$
\begin{aligned}
\left(\frac{x}{3}+\frac{1}{x}\right)^5 &= {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 \\
&\quad + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5 \\
&= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5} \\
&= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}
\end{aligned}
$$

**Question** 5:

Expand $\left(x+\frac{1}{x}\right)^6$

**Answer**:

By using Binomial Theorem, the expression $\left(x+\frac{1}{x}\right)^6$ can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2$$
$$+ {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$
$$= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6}$$
$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

**Question** 6:

Using Binomial Theorem, evaluate $(96)^3$

**Answer**:

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 96 = 100 – 4

$$\therefore (96)^3 = (100-4)^3$$
$$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - {}^3C_3(4)^3$$
$$= (100)^3 - 3(100)^2(4) + 3(100)(4)^2 - (4)^3$$
$$= 1000000 - 120000 + 4800 - 64$$
$$= 884736$$

**Question** 7:

Using Binomial Theorem, evaluate $(102)^5$

**Answer**:

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 102 = 100 + 2

$$\therefore (102)^5 = (100+2)^5$$
$$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$$
$$= (100)^5 + 5(100)^4(2) + 10(100)^3(2)^2 + 10(100)^2(2)^3 + 5(100)(2)^4 + (2)^5$$
$$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$$
$$= 11040808032$$

**Question** 8:

Using Binomial Theorem, evaluate $(101)^4$

**Answer**:

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 101 = 100 + 1

$$\therefore (101)^4 = (100+1)^4$$
$$= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4$$
$$= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4$$
$$= 100000000 + 4000000 + 60000 + 400 + 1$$
$$= 104060401$$

**Question** 9:

Using Binomial Theorem, evaluate $(99)^5$

**Answer**:

99 can be written as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 99 = 100 – 1

$$\therefore (99)^5 = (100-1)^5$$
$$= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5$$
$$= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1$$
$$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$$
$$= 10010000500 - 500100001$$
$$= 9509900499$$

**Question** 10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

**Answer**:

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ can be obtained as

$$(1.1)^{10000} = (1+0.1)^{10000}$$
$$= {}^{10000}C_0 + {}^{10000}C_1(1.1) + \text{Other positive terms}$$
$$= 1 + 10000 \times 1.1 + \text{Other positive terms}$$
$$= 1 + 11000 + \text{Other positive terms}$$
$$> 1000$$

Hence, $(1.1)^{10000} > 1000$

**Question** 11:

Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$ .

**Answer**:

Using Binomial Theorem, the expressions, $(a + b)^4$ and $(a - b)^4$, can be expanded as

$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$
$$(a-b)^4 = {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4$$
$$\therefore (a+b)^4 - (a-b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$
$$-\left[{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4\right]$$
$$= 2\left({}^4C_1a^3b + {}^4C_3ab^3\right) = 2\left(4a^3b + 4ab^3\right)$$
$$= 8ab\left(a^2 + b^2\right)$$

By putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4 = 8\left(\sqrt{3}\right)\left(\sqrt{2}\right)\left\{\left(\sqrt{3}\right)^2 + \left(\sqrt{2}\right)^2\right\}$$
$$= 8\left(\sqrt{6}\right)\{3+2\} = 40\sqrt{6}$$

**Question** 12:

Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$.

**Answer**:

Using Binomial Theorem, the expressions, $(x + 1)^6$ and $(x - 1)^6$, can be expanded as

$$(x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6$$
$$(x-1)^6 = {}^6C_0x^6 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 + {}^6C_4x^2 - {}^6C_5x + {}^6C_6$$
$$\therefore (x+1)^6 + (x-1)^6 = 2\left[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6\right]$$
$$= 2\left[x^6 + 15x^4 + 15x^2 + 1\right]$$

By putting $x = \sqrt{2}$, we obtain

$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 2\left[\left(\sqrt{2}\right)^6 + 15\left(\sqrt{2}\right)^4 + 15\left(\sqrt{2}\right)^2 + 1\right]$$
$$= 2\left(8 + 15\times4 + 15\times2 + 1\right)$$
$$= 2\left(8 + 60 + 30 + 1\right)$$
$$= 2(99) = 198$$

**Question** 13:

Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever *n* is a positive integer.

**Answer**:

In order to show that $9^{n+1}-8n-9$ is divisible by 64, it has to be proved that,

$9^{n+1}-8n-9=64k$ , where *k* is some natural number

By Binomial Theorem,

$$(1+a)^m = {}^mC_0 + {}^mC_1a + {}^mC_2a^2 + \ldots + {}^mC_ma^m$$

For *a* = 8 and *m* = *n* + 1, we obtain

$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \ldots + {}^{n+1}C_{n+1}(8)^{n+1}$$
$$\Rightarrow 9^{n+1} = 1+(n+1)(8)+8^2\left[{}^{n+1}C_2 + {}^{n+1}C_3\times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$
$$\Rightarrow 9^{n+1} = 9+8n+64\left[{}^{n+1}C_2 + {}^{n+1}C_3\times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$
$$\Rightarrow 9^{n+1}-8n-9 = 64k,\ \text{where } k = {}^{n+1}C_2 + {}^{n+1}C_3\times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1} \text{ is a natural number}$$

Thus,

$9^{n+1}-8n-9$ is divisible by 64, whenever *n* is a positive integer.

**Question** 14:

Prove that $\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$ .

**Answer**:

By Binomial Theorem,

$$\sum_{r=0}^{n} {}^nC_r\, a^{n-r}b^r = (a+b)^n$$

By putting *b* = 3 and *a* = 1 in the above equation, we obtain

$$\sum_{r=0}^{n} {}^nC_r\,(1)^{n-r}(3)^r = (1+3)^n$$
$$\Rightarrow \sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$$

Hence, proved.

**Question** 4:

If *a* and *b* are distinct integers, prove that *a* – *b* is a factor of *an* – *bn*, whenever *n* is a positive integer.

[**Hint:** write *an* = (*a* – *b* + *b*)*n* and expand]

**Answer**:

In order to prove that (*a* – *b*) is a factor of (*an* – *bn*), it has to be proved that

*an* – *bn* = *k* (*a* – *b*), where *k* is some natural number

It can be written that, *a* = *a* – *b* + *b*

$$\therefore a^n = (a-b+b)^n = \left[(a-b)+b\right]^n$$
$$= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + {}^nC_n b^n$$
$$= (a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + b^n$$
$$\Rightarrow a^n - b^n = (a-b)\left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right]$$
$$\Rightarrow a^n - b^n = k(a-b)$$
$$\text{where, } k = \left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right] \text{ is a natural number}$$

This shows that (*a* – *b*) is a factor of (*an* – *bn*), where *n* is a positive integer.

**Question** 5:

Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$.

**Answer**:

Firstly, the expression $(a + b)^6 - (a - b)^6$ is simplified by using Binomial Theorem.

This can be done as

$$(a+b)^6 = {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6$$
$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$
$$(a-b)^6 = {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5a^1b^5 + {}^6C_6b^6$$
$$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$
$$\therefore (a+b)^6 - (a-b)^6 = 2\left[6a^5b + 20a^3b^3 + 6ab^5\right]$$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6 = 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right) + 20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3 + 6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]$$
$$= 2\left[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}\right]$$
$$= 2 \times 198\sqrt{6}$$
$$= 396\sqrt{6}$$

**Question** 6:

Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4$ .

**Answer**:

Firstly, the expression $(x + y)^4 + (x - y)^4$ is simplified by using Binomial Theorem.

This can be done as

$$(x+y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x-y)^4 = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4$$
$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$
$$\therefore (x+y)^4 + (x-y)^4 = 2\left(x^4 + 6x^2y^2 + y^4\right)$$

Putting $x = a^2$ and $y = \sqrt{a^2-1}$, we obtain

$$\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4 = 2\left[\left(a^2\right)^4 + 6\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2 + \left(\sqrt{a^2-1}\right)^4\right]$$
$$= 2\left[a^8 + 6a^4\left(a^2-1\right) + \left(a^2-1\right)^2\right]$$
$$= 2\left[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1\right]$$
$$= 2\left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1\right]$$
$$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$$

**Question** 7:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

**Answer**:

0.99 = 1 – 0.01

$$
\begin{aligned}
\therefore (0.99)^5 &= (1-0.01)^5 \\
&= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2 \qquad \text{(Approximately)} \\
&= 1 - 5(0.01) + 10(0.01)^2 \\
&= 1 - 0.05 + 0.001 \\
&= 1.001 - 0.05 \\
&= 0.951
\end{aligned}
$$

Thus, the value of $(0.99)^5$ is approximately 0.951

## Miscellaneous Exercise

**Question** 1:

If *a* and *b* are distinct integers, prove that *a* – *b* is a factor of *an* – *bn*, whenever *n* is a positive integer.

[**Hint:** write *an* = (*a* – *b* + *b*)*n* and expand]

**Answer**:

In order to prove that (*a* – *b*) is a factor of (*an* – *bn*), it has to be proved that

*an* – *bn* = *k* (*a* – *b*), where *k* is some natural number

It can be written that, *a* = *a* – *b* + *b*

$$
\begin{aligned}
\therefore a^n &= (a-b+b)^n = \left[(a-b)+b\right]^n \\
&= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + {}^nC_nb^n \\
&= (a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + b^n \\
\Rightarrow a^n - b^n &= (a-b)\left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right] \\
\Rightarrow a^n - b^n &= k(a-b)
\end{aligned}
$$

$$\text{where, } k = \left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right] \text{ is a natural number}$$

This shows that (*a* – *b*) is a factor of (*an* – *bn*), where *n* is a positive integer.

**Question** 2:

Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$ .

**Answer**:

Firstly, the expression $(a+b)^6-(a-b)^6$ is simplified by using Binomial Theorem.

This can be done as

$$(a+b)^6 = {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6$$
$$= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$
$$(a-b)^6 = {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5a^1b^5 + {}^6C_6b^6$$
$$= a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$$
$$\therefore (a+b)^6-(a-b)^6 = 2\left[6a^5b+20a^3b^3+6ab^5\right]$$

Putting $a=\sqrt{3}$ and $b=\sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6 = 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right)+20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3+6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]$$
$$=2\left[54\sqrt{6}+120\sqrt{6}+24\sqrt{6}\right]$$
$$=2\times198\sqrt{6}$$
$$=396\sqrt{6}$$

**Question** 3:

Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$ .

**Answer**:

Firstly, the expression $(x+y)^4+(x-y)^4$ is simplified by using Binomial Theorem.

This can be done as

$$(x+y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x-y)^4 = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4$$
$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$
$$\therefore (x+y)^4 + (x-y)^4 = 2\left(x^4 + 6x^2y^2 + y^4\right)$$

Putting $x = a^2$ and $y = \sqrt{a^2-1}$, we obtain

$$\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4 = 2\left[\left(a^2\right)^4 + 6\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2 + \left(\sqrt{a^2-1}\right)^4\right]$$
$$= 2\left[a^8 + 6a^4\left(a^2-1\right) + \left(a^2-1\right)^2\right]$$
$$= 2\left[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1\right]$$
$$= 2\left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1\right]$$
$$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$$

**Question** 4:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

**Answer**:

0.99 = 1 – 0.01

$$\therefore (0.99)^5 = (1-0.01)^5$$
$$= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2 \qquad \text{(Approximately)}$$
$$= 1 - 5(0.01) + 10(0.01)^2$$
$$= 1 - 0.05 + 0.001$$
$$= 1.001 - 0.05$$
$$= 0.951$$

Thus, the value of $(0.99)^5$ is approximately 0.951.

**Question** 5:

Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$.

**Answer**:

Using Binomial Theorem, the given expression $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$ can be expanded as

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$= {}^4C_0\left(1+\frac{x}{2}\right)^4 - {}^4C_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4$$

$$= \left(1+\frac{x}{2}\right)^4 - 4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$= \left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$$

$$= \left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \quad ...(1)$$

Again by using Binomial Theorem, we obtain

$$\left(1+\frac{x}{2}\right)^4 = {}^4C_0(1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)^1\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$$

$$= 1 + 4\times\frac{x}{2} + 6\times\frac{x^2}{4} + 4\times\frac{x^3}{8} + \frac{x^4}{16}$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \quad ...(2)$$

$$\left(1+\frac{x}{2}\right)^3 = {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$$

$$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$$

**Question** 6:

Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

**Answer**:

Using Binomial Theorem, the given expression $(3x^2 - 2ax + 3a^2)^3$ can be expanded as

$$\left[(3x^2 - 2ax) + 3a^2\right]^3$$
$$= {}^3C_0 (3x^2 - 2ax)^3 + {}^3C_1 (3x^2 - 2ax)^2 (3a^2) + {}^3C_2 (3x^2 - 2ax)(3a^2)^2 + {}^3C_3 (3a^2)^3$$
$$= (3x^2 - 2ax)^3 + 3(9x^4 - 12ax^3 + 4a^2x^2)(3a^2) + 3(3x^2 - 2ax)(9a^4) + 27a^6$$
$$= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^3 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6$$
$$= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \quad ...(1)$$

Again by using Binomial Theorem, we obtain

$$(3x^2 - 2ax)^3$$
$$= {}^3C_0 (3x^2)^3 - {}^3C_1 (3x^2)^2 (2ax) + {}^3C_2 (3x^2)(2ax)^2 - {}^3C_3 (2ax)^3$$
$$= 27x^6 - 3(9x^4)(2ax) + 3(3x^2)(4a^2x^2) - 8a^3x^3$$
$$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 \quad ...(2)$$

From (1) and (2), we obtain

$$(3x^2 - 2ax + 3a^2)^3$$
$$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$
$$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$$